"STERADIANS" AND SPHERICAL EXCESS

By PROFESSOR GEORGE W. EVANS
Charleston High School, Boston.

In elementary geometry, or oftener in trigonometry, we speak of radian measure of plane angles; but, if we ever mention the measure of a solid angle by the included area of a unit sphere, it is a mere comment, and seems to have nothing to do with the fact that the area of a spherical triangle is proportional to its spherical excess. It is not easy for the pupil to infer, and he generally does not infer, that the spherical excess, expressed in radians, is precisely this measure of the solid angle, and, if multiplied by r^2, gives the area of the triangle.

Quite as mysterious is the way in which the perimenter of the polar triangle appears opportunely to help us out with the spherical excess, coming from a dimly remembered time when we were talking about the face angles of a polyedral, and vanishing again into a useless attic.

What I am now suggesting is a method of connecting up these loose ends, and of extending the results to the areas of convex polygons and circles on a sphere, and to the measure, in "steradians," of convex polyedral angles and right circular cones. A steradian is defined as the solid angle which, having its vertex at the center of a sphere of radius r, intercepts the area r^2 on the surface of that sphere.

Polar Polygons

In any sphere call the center V and any spherical polygon $ABCDE$ . . . : then the polar polygon $A'B'C'D'E'$ . . . . is determined by drawing radii VA' perpendicular to VBC, VB' to VCD, and so on. If these perpendiculars are all drawn outwards with respect to the polyedral angle $V\text{-}ABCDE$ . . ., it is easy to prove that the angle $A'VB'$ is supplementary to the diedral $B\text{-}VC\text{-}D$, $B'VC'$ *to* $C\text{-}VD\text{-}E$, and so on, around the polygon. This construction includes triangles, but picks out the polar triangle opposite to the one we have been used to consider.

The order of naming vertices as given in the preceding paragraph seems whimsical: it serves, however, to indicate the correspondence of sides and angles in the usual way when the

polygon degenerates to a triangle. A much more satisfactory method would be to name one of the polygons by its sides, thus: $abcde$ . . . and its polar polygon by its vertices $A'B'C'D'E'$ . . ., with the understanding that the radius VA' is perpendicular to the plane of the arc a at the point V.

For example, the triangle abc has for its polar triangle $A'B'C'$, such that $VA' \perp a$, $VB' \perp b$, $VC' \perp c$. Since $VA' \perp a$, and $VB' \perp b$, the plane $VA'B'$ is perpendicular to the edge of the diedral aVB; consequently the intersection of the arcs a and b (the point ab) is the pole of the arc $A'B'$, and the triangle abc is the polar triangle of $A'B'C'$.

These proofs of the usual theorems about polar triangles are, at the worst, not more difficult than the usual proofs; and they can readily be extended to polygons, if, indeed, it is not quite as convenient to give proofs about polygons in general at the start.

Spherical Excess of a Polygon

Let $ABCDE$ . . . . be any convex spherical polygon, and $a'b'c'd'e'$ . . . . its polar polygon. The planes of the sides of each of these polygons form a polyedral angle with vertex at V, the center of the sphere. What can we discover about the sum of the angles $A\ B\ C\ D\ E$ . . . ?

We may be able to use the fact, proved in the next paragraph, that $A = \pi - a'$, and the further fact that a' is the measure of one of the face angles of the polyedral angle belonging to a'b'c'd'e' . . . .

Draw a plane to cut all the lateral edges of the polyedral angle of $a'b'c'd'e'$ . . ., and thereby form a pyramid. In the face of this pyramid that contains arc a', not only is it true that the angle at $V = a'$, but also that the base angles of the triangle are together equal to A; that is $A = \pi - a'$. Then, if we represent Σ the sum $A + B + C + D + E +$ . . . and by p' the sum $a' + b' + c' + d' + e' +$ . . . we have the equation (1) $\Sigma = n\pi - p'$.

On the other hand, looking at the base of the pyramid, any angle of that plane polygon is less than the sum of the two angles of the lateral faces that come next to it; and consequently the

sum of all the angles of the base polygon, which is $(n - 2)\pi$, will be less than $\sum$ by some undetermined amount which we may represent by X. Then we have the equation

$$(2)\ X = \sum - (n - 2)\pi$$

From these two equations we find

$$(3)\ X = 2\pi - p'$$

On equation (2) we may base the definition:

The spherical excess of a convex spherical polygon is the amount by which the sum of its angles exceeds the sum of the angles of a plane polygon of the same number of sides.

And we may state equation (3) as a theorem:

The spherical excess of a convex spherical polygon is equal to the difference in radians between the perimeter of its polar polygon and the circumference of a great circle.

The Area of a Spherical Triangle

Expressing the angles of a spherical triangle ABC in radians, its area by S, and representing by T_1, T_2 and T_3, respectively, the areas of the three triangles that piece S out so as to make lunes with angles A, B, and C, we have the equations:

$$S + T_1 = \frac{A}{2\pi} 4\pi r^2 = 2\ Ar^2$$
$$S + T_2 = 2Br^2$$
$$S + T_3 = 2Cr^2$$
$$3S + T_1 + T_2 + T_3 = 2(A + B + C)r^2$$
$$2S + 2\pi r = 2(A + B + C)r^2$$
$$S = (A + B + C - \pi)r^2$$

Or, representing the spherical excess of the spherical triangle by X,

$$S = Xr^2$$

The Area of a Spherical Polygon

In the convex spherical polygon $ABCDE$ . . . . draw all possible diagonals through the vertex A. This will divide the polygon into triangles having a common vertex A. Every side of the polygon except the two that pass through A, will identify a triangle by serving as its base; there will conseqently be n-2 triangles. Now if we use the same suffix for all the angles of one

triangle, we can get along with one letter for each angle; and we can write for the areas of the successive triangles—

$$S_1 = (A_1 + B_1 + C_1 - \pi)r^2 = X_1r^2$$
$$S_2 = (A_2 + C_2 + D_2 - \pi)r^2 = X_2r^2$$
$$S_3 = (A_3 + D_3 + E_3 - \pi)r^2 = X_3r^2$$

and so on; and for the area of the polygon—

$$S = S_1 + S_2 + S_3 + \ldots .$$
$$S = (A + B + C_1 + C_2 + D_2 + D_3 + E_3 + E_4 + \ldots . - (n - 2)\pi)r^2$$
$$S = A + B + C + D + E + \ldots - (n - 2)\pi)r^2$$
$$S = Xr^2$$

Hence the theorem:

The area of any spherical polygon is equal to its spherical excess, in radians, multiplied by the square of the radius.

The Area of a Small Circle on a Sphere

We may consider the area enclosed by a small circle on a sphere as the limit of the area of an equilateral spherical polygon of n sides, inscribed in that circle, as n is indefinitely increased. Every one of those equilateral spherical polygons will have a polar polygon, which will be inscribed in a small circle described from the nearer pole of the given circle with a polar distance $\theta + \frac{\pi}{2}$, where θ is the polar distance of the given circle. This we may call the polar circle of the given circle; and each element in the central cone determined by either circle will be perpendicular to the corresponding element in the other cone and in the same axial plane with it.

The circumference of the given circle will be $2\pi r \sin\theta$ and of the polar circle $2\pi r \cos\theta$.

For any one of the equilateral polygons inscribed in the given circle we have $S = (2\pi - p')\, r^2$ where p' is the perimeter of the polar polygon, expressed in radians.

The limit of p', as n increases, is

$$\frac{2\pi r \cos\theta}{r} = 2\pi \cos\theta$$

The area of the circle, which we may represent by Z is the limit of $(2\pi - p^1)r^2$, that is:

$$Z = 2\pi r^2 - 2\pi r^2 \cos\theta$$

$$= 2\pi\ (1 - \cos\theta)\ r^2 = 4\pi(\sin^2 \frac{\theta}{2})\ r^2$$

We have, however, as a formula for Z, independently arrived at, $Z = 2\pi rh$, where h is the altitude of the cone. It is therefore reassuring to notice that

$$h = r(1 - \cos\theta) = 2r\sin^2 \frac{\theta}{2}$$